

BLOCKS OF THE GROTHENDIECK RING OF EQUIVARIANT BUNDLES ON A FINITE GROUP

by

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Abstract. — If G is a finite group, the Grothendieck group $\mathbf{K}_G(G)$ of the category of G -equivariant \mathbb{C} -vector bundles on G (for the action of G on itself by conjugation) is endowed with a structure of (commutative) ring. If K is a sufficiently large extension of \mathbb{Q}_p and \mathcal{O} denotes the integral closure of \mathbb{Z}_p in K , the K -algebra $K\mathbf{K}_G(G) = K \otimes_{\mathbb{Z}} \mathbf{K}_G(G)$ is split semisimple. The aim of this paper is to describe the \mathcal{O} -blocks of the \mathcal{O} -algebra $\mathcal{O}\mathbf{K}_G(G)$.

1. Notation, introduction

1.A. Groups. — We fix in this paper a finite group G , a prime number p and a finite extension K of the p -adic field \mathbb{Q}_p such that KH is split for all subgroups H of G . We denote by \mathcal{O} the integral closure of \mathbb{Z}_p in K , by \mathfrak{p} the maximal ideal of \mathcal{O} , by k the residue field of \mathcal{O} (i.e. $k = \mathcal{O}/\mathfrak{p}$). We denote by $\text{Irr}(KG)$ the set of irreducible characters of G (over K).

A p -element (respectively p' -element) of G is an element whose order is a power of p (respectively prime to p). If $g \in G$, we denote by g_p and $g_{p'}$ the unique elements of G such that $g = g_p g_{p'} = g_{p'} g_p$, g_p is a p -element and $g_{p'}$ is a p' -element. The set of p -elements (respectively p' -elements) of G is denoted by G_p (respectively $G_{p'}$).

If X is a G -set (i.e. a set endowed with a left G -action), we denote by $[G \backslash X]$ a set of representatives of G -orbits in X . The reader can check that we will use formulas like

$$\sum_{x \in [G \backslash X]} f(x)$$

(or families like $(f(x))_{x \in [G \backslash X]}$) only whenever $f(x)$ does not depend on the choice of the representative x in its G -orbit. If X is a set- G (i.e. a set endowed with a right G -action), we define similarly $[X/G]$ and will use it according to the same principles.

1.B. Blocks. — A *block idempotent* of kG (respectively $\mathcal{O}G$) is a primitive idempotent of the center $Z(kG)$ (respectively $Z(\mathcal{O}G)$) of $\mathcal{O}G$. We denote by $\text{Blocks}(kG)$ (respectively $\text{Blocks}(\mathcal{O}G)$) the set of block idempotents of kG (respectively $\mathcal{O}G$). Reduction modulo \mathfrak{p} induces a bijection $\text{Blocks}(\mathcal{O}G) \xrightarrow{\sim} \text{Blocks}(kG)$, $e \mapsto \bar{e}$ (whose inverse is denoted by $e \mapsto \tilde{e}$).

A *p-block* of G is a subset \mathcal{B} of $\text{Irr}(G)$ such that $\mathcal{B} = \text{Irr}(KGe)$, for some block idempotent e of $\mathcal{O}G$.

1.C. Fourier coefficients. — Let

$$\text{IrrPairs}(G) = \{(g, \gamma) \mid g \in G \text{ and } \gamma \in \text{Irr}(KC_G(g))\}$$

and

$$\text{BlPairs}_p(G) = \{(s, e) \mid s \in G_{p'} \text{ and } e \in \text{Blocks}(\mathcal{O}C_G(s))\}.$$

The group G acts (on the left) on these two sets by conjugation. We set

$$\mathcal{M}(G) = [G \backslash \text{IrrPairs}(G)] \quad \text{and} \quad \mathcal{M}^p(G) = [G \backslash \text{BlPairs}_p(G)].$$

If $(g, \gamma), (h, \eta) \in \text{IrrPairs}(G)$, we define, following Lusztig [Lu, 2.5(a)],

$$\{(g, \gamma), (h, \eta)\} = \frac{1}{|C_G(g)| \cdot |C_G(h)|} \sum_{\substack{x \in G \\ xhx^{-1} \in C_G(g)}} \gamma(xhx^{-1})\eta(x^{-1}g^{-1}x).$$

Note that $\{(g, \gamma), (h, \eta)\}$ depends only on the G -orbit of (g, γ) and on the G -orbit of (h, η) .

1.D. Vector bundles. — Except from Proposition 2.3 below, all the definitions, all the results in this subsection can be found in [Lu, §2]. We denote by $\mathcal{Bun}_G(G)$ the category of G -equivariant finite dimensional K -vector bundles on G (for the action of G by conjugation). Its Grothendieck group $\mathbf{K}_G(G)$ is endowed with a ring structure. For each $(g, \gamma) \in \mathcal{M}(G)$, let $V_{g, \gamma}$ be the isomorphism class (in $\mathbf{K}_G(G)$) of the simple object in $\mathcal{Bun}_G(G)$ associated with (g, γ) , as in [Lu, §2.5] (it is denoted $U_{g, \gamma}$ there). Then

$$\mathbf{K}_G(G) = \bigoplus_{(g, \gamma) \in \mathcal{M}(G)} \mathbb{Z} V_{g, \gamma}.$$

The K -algebra $K\mathbf{K}_G(G) = K \otimes_{\mathbb{Z}} \mathbf{K}_G(G)$ is split semisimple and commutative. Its simple modules (which have dimension one) are also parametrized by $\mathcal{M}(G)$: if $(g, \gamma) \in \mathcal{M}(G)$, the K -linear map

$$\Psi_{g, \gamma} : K\mathbf{K}_G(G) \longrightarrow K$$

defined by

$$\Psi_{g, \gamma}(V_{h, \eta}) = \frac{|C_G(g)|}{\gamma(1)} \cdot \{(h^{-1}, \eta), (g, \gamma)\}$$

is a morphism of K -algebras and all morphisms of K -algebras $K\mathbf{K}_G(G) \rightarrow K$ are obtained in this way.

We define similarly block idempotents of $k\mathbf{K}_G(G)$ and $\mathcal{O}\mathbf{K}_G(G)$, as well as p -blocks of $\mathcal{M}(G) \xrightarrow{\sim} \text{Irr}(K\mathbf{K}_G(G))$.

1.E. Brauer maps. — Let Λ denote one of the two rings \mathcal{O} or k . If $g \in G$ (and if we set $s = g_{p'}$), we denote by Br_g^Λ the Λ -linear map

$$\text{Br}_g^\Lambda : \Lambda C_G(s) \rightarrow \Lambda C_G(g)$$

such that

$$\text{Br}_g^\Lambda(h) = \begin{cases} h & \text{if } h \in C_G(g), \\ 0 & \text{if } h \notin C_G(g), \end{cases}$$

for all $h \in C_G(s)$. Recall [Is, Lemma 15.32] that

$$(1.1) \quad \text{Br}_g^k \text{ induces a morphism of algebras } Z(kC_G(s)) \rightarrow Z(kC_G(g)).$$

Therefore, if $e \in \text{Blocks}(\mathcal{O}C_G(s))$, then $\text{Br}_g^k(e)$ is an idempotent of $Z(kC_G(g))$ (possibly equal to zero) and we can write it a sum $\text{Br}_g^k(e) = e_1 + \dots + e_n$, where e_1, \dots, e_n are pairwise distinct block idempotents of $kC_G(g)$. We then set

$$\beta_g^\mathcal{O}(e) = \sum_{i=1}^n \tilde{e}_i.$$

It is an idempotent (possibly equal to zero, possibly non-primitive) of $Z(\mathcal{O}C_G(g))$.

1.F. The main result. — In order to state more easily our main result, it will be more convenient (though it is not strictly necessary) to fix a particular set of representatives of conjugacy classes of G .

Hypothesis and notation. From now on, and until the end of this paper, we denote by:

- $[G_{p'}/\sim]$ a set of representatives of conjugacy classes of p' -elements in G .
- $[G/\sim]$ a set of representatives of conjugacy classes of elements of G such that, for all $g \in [G/\sim]$, $g_{p'} \in [G_{p'}/\sim]$.

We also assume that, if $(g, \gamma) \in \mathcal{M}(G)$ or $(s, e) \in \mathcal{M}^p(G)$, then $g \in [G/\sim]$ and $s \in [G_{p'}/\sim]$.

If $(s, e) \in \mathcal{M}^p(G)$, we define $\mathcal{B}_G(s, e)$ to be the set of pairs $(g, \gamma) \in \mathcal{M}(G)$ such that:

- (1) $g_{p'} = s$.
- (2) $\gamma \in \text{Irr}(KC_G(g)\beta_g^\mathcal{O}(e))$.

Theorem 1.2. — *The map $(s, e) \mapsto \mathcal{B}_G(s, e)$ induces a bijection between $\mathcal{M}^p(G)$ to the set of p -blocks of $\mathcal{M}(G)$.*

2. Proof of Theorem 1.2

2.A. Central characters and congruences. — If $(g, \gamma) \in \text{IrrPairs}(G)$, we denote by $\omega_{g, \gamma} : Z(KC_G(g)) \rightarrow K$ the *central character* associated with γ (if $z \in Z(KC_G(g))$, then $\omega_{g, \gamma}(z)$ is the scalar through which z acts on an irreducible $KC_G(g)$ -module affording the character γ). It is a morphism of algebras: when restricted to $Z(\mathcal{O}C_G(g))$, it has values in \mathcal{O} .

If $h \in C_G(g)$, we denote by $\Sigma_g(h)$ conjugacy class of h in $C_G(g)$ and we set

$$\hat{\Sigma}_g(h) = \sum_{v \in \Sigma_g(h)} v \in Z(\mathcal{O}C_G(g)).$$

We have

$$(2.1) \quad \omega_{g, \gamma}(\hat{\Sigma}_g(h)) = \frac{|\Sigma_g(h)| \cdot \gamma(h)}{\gamma(1)}.$$

We also recall the following classical results:

Proposition 2.2. — *If $g \in G$ and γ, γ' are two irreducible characters of $C_G(g)$, then γ and γ' lie in the same p -block of $C_G(g)$ if and only if*

$$\omega_{g, \gamma}(\hat{\Sigma}_g(h)) \equiv \omega_{g, \gamma'}(\hat{\Sigma}_g(h)) \pmod{\mathfrak{p}}$$

for all $h \in C_G(g)$.

Proposition 2.3. — *Let (g, γ) and (g', γ') be two elements of $\mathcal{M}(G)$. Then (g, γ) and (g', γ') belong to the same p -block of $\mathcal{M}(G)$ if and only if*

$$\Psi_{g, \gamma}(V_{h, \eta}) \equiv \Psi_{g', \gamma'}(V_{h, \eta}) \pmod{\mathfrak{p}}$$

for all $(h, \eta) \in \mathcal{M}(G)$.

2.B. Around the Brauer map. — As Brauer maps are morphisms of algebras, we have

$$\sum_{e \in \text{Blocks}(kC_G(g_{p'}))} \text{Br}_g^p(e) = 1,$$

and so

$$(2.4) \quad \text{The family } (\mathcal{B}_G(g, e))_{(g, e) \in \mathcal{M}^p(G)} \text{ is a partition of } \mathcal{M}(G).$$

Now, let $(g, \gamma) \in \mathcal{M}(G)$ and let $s = g_{p'}$. If $e \in \text{Blocks}(\mathcal{O}C_G(s))$ is such that $\gamma \in \text{Irr}(KC_G(g)\beta_g^\theta(e))$, and if $\sigma \in \text{Irr}(KC_G(s)e)$, then [Is, Lemma 15.44]

$$(2.5) \quad \omega_{s,\sigma}(z) \equiv \omega_{g,\gamma}(\text{Br}_g^\theta(z)) \pmod{\mathfrak{p}}$$

for all $z \in Z(\mathcal{O}C_G(s))$.

2.C. Rearranging the formula for $\Psi_{g,\gamma}$. — If $(g, \gamma), (h, \eta) \in \text{IrrPairs}(g)$ then

$$(2.6) \quad \Psi_{g,\gamma}(V_{h,\eta}) = \sum_{\substack{x \in [C_G(g) \backslash G / C_G(h)] \\ xhx^{-1} \in C_G(g)}} \eta(x^{-1}gx) \omega_{g,\gamma}(\hat{\Sigma}_g(xhx^{-1})).$$

Proof. — By definition,

$$\Psi_{g,\gamma}(V_{h,\eta}) = \frac{1}{\gamma(1) \cdot |C_G(h)|} \sum_{\substack{x \in G \\ xhx^{-1} \in C_G(g)}} \eta(x^{-1}gx) \gamma(xhx^{-1}) = \frac{1}{\gamma(1)} \sum_{\substack{x \in [G/C_G(h)] \\ xhx^{-1} \in C_G(g)}} \eta(x^{-1}gx) \gamma(xhx^{-1}).$$

Now, if $x \in G$ is such that $xhx^{-1} \in C_G(g)$ and if $u \in C_G(g)$, then

$$\eta((ux)^{-1}g(ux)) \gamma((ux)h(ux)x^{-1}) = \eta(x^{-1}gx) \gamma(xhx^{-1}).$$

So we can gather the terms in the last sum according to their $C_G(g)$ -orbit. We get

$$\Psi_{g,\gamma}(V_{h,\eta}) = \sum_{\substack{x \in [C_G(g) \backslash G / C_G(h)] \\ xhx^{-1} \in C_G(g)}} \eta(x^{-1}gx) \frac{|C_G(g)|}{|C_G(g) \cap xC_G(h)x^{-1}|} \cdot \frac{\gamma(xhx^{-1})}{\gamma(1)}.$$

But, for x in G such that $xhx^{-1} \in C_G(g)$,

$$\frac{|C_G(g)|}{|C_G(g) \cap xC_G(h)x^{-1}|} = |\Sigma_g(xhx^{-1})|,$$

so the result follows from 2.1. □

Corollary 2.7. — Let $g \in [G/\sim]$ and let $\gamma, \gamma' \in \text{Irr}(KC_G(g))$ lying in the same p -block of $C_G(g)$. Then (g, γ) and (g, γ') lie in the same p -block of $\mathcal{M}(G)$.

Proof. — This follows from 2.6 and Proposition 2.3. □

2.D. p' -part. — Fix $(g, \gamma) \in \mathcal{M}(G)$. Then it follows from 2.6 that, for all $\chi \in \text{Irr}(KG)$,

$$(2.8) \quad \Psi_{g, \gamma}(V_{1, \chi}) = \chi(g).$$

Proposition 2.9. — *Let (g, γ) and (h, η) be two elements in $\mathcal{M}(G)$ which lie in the same p -block. Then $g_{p'} = h_{p'}$.*

Proof. — By Proposition 2.3 and Equality 2.8, it follows from the hypothesis that

$$\chi(g) \equiv \chi(h) \pmod{p}$$

for all $\chi \in \text{Irr}(KG)$. Hence $g_{p'}$ and $h_{p'}$ are conjugate in G (see [Bo, Proposition 2.14]), so they are equal according to our conventions explained in §1.F. \square

Proposition 2.10. — *Let $s \in G_{p'}$ and let $\sigma, \sigma' \in \text{Irr}(KC_G(s))$. Then (s, σ) and (s, σ') lie in the same p -block if and only if σ and σ' lie in the same p -block of $C_G(s)$.*

Proof. — The if part has been proved in Corollary 2.7. Conversely, assume that (s, σ) and (s, σ') lie in the same p -block. Fix $h \in C_G(s)$. Then $s \in C_G(h)$. Let $\eta_{s, h} : C_G(h) \rightarrow K$ be the class function on $C_G(h)$ defined by:

$$\eta_{s, h}(g) = \begin{cases} 1 & \text{if } g_{p'} \text{ and } s \text{ are conjugate in } C_G(h), \\ 0 & \text{otherwise.} \end{cases}$$

It follows from [Bo, Proposition 2.20] that $\eta_{s, h} \in \mathcal{O} \text{Irr}(KC_G(h))$. Therefore, by 2.6 and Proposition 2.3,

$$(\#) \quad \sum_{\substack{x \in [C_G(s) \backslash G / C_G(h)] \\ xhx^{-1} \in C_G(s)}} \eta_{s, h}(x^{-1}sx) \left(\omega_{s, \sigma}(\hat{\Sigma}_g(xhx^{-1})) - \omega_{s, \sigma'}(\hat{\Sigma}_g(xhx^{-1})) \right) \equiv 0 \pmod{p}.$$

Now, let $x \in G$ be such that $xhx^{-1} \in C_G(s)$. Since $x^{-1}sx$ is also a p' -element, $\eta_{s, h}(x^{-1}sx) = 1$ if and only if s and $x^{-1}sx$ are conjugate in $C_G(h)$ that is, if and only if $x \in C_G(s)C_G(h)$. So it follows from (#) that

$$\omega_{s, \sigma}(\hat{\Sigma}_g(h)) \equiv \omega_{s, \sigma'}(\hat{\Sigma}_g(h)) \pmod{p}$$

for all $h \in C_G(s)$. This shows that σ and σ' lie in the same p -block of $C_G(s)$. \square

2.E. Last step. — We shall prove here the last intermediate result:

Proposition 2.11. — *Let $(s, e) \in \mathcal{M}^p(G)$ and let $(g, \gamma), (g', \gamma') \in \mathcal{B}_G(s, e)$. Then (g, γ) and (g', γ') are in the same p -block of $\mathcal{M}(G)$.*

Proof. — We fix $\sigma \in \text{Irr}(KC_G(s)e)$. It is sufficient to show that (g, γ) and (s, σ) are in the same p -block of $\mathcal{M}(G)$. For this, let $(h, \eta) \in \mathcal{M}(G)$. By Proposition 2.9, we have $g_{p'} = s$, so $C_G(g) \subset C_G(s)$. So 2.6 can be rewritten:

$$\Psi_{g, \gamma}(V_{h, \eta}) = \sum_{x \in [C_G(s) \backslash G / C_G(h)]} \sum_{\substack{y \in [C_G(g) \backslash C_G(s) x C_G(h) / C_G(h)] \\ y h y^{-1} \in C_G(g)}} \eta(y^{-1} g y) \omega_{g, \gamma}(\hat{\Sigma}_g(y h y^{-1})).$$

Now, let $x \in [C_G(s) \backslash G / C_G(h)]$ and $y \in [C_G(g) \backslash C_G(s) x C_G(h) / C_G(h)]$ be such that $y h y^{-1} \in C_G(g)$. Then $y h y^{-1} \in C_G(s)$ and so $x h x^{-1} \in C_G(s)$. Moreover $y^{-1} s y$ is conjugate to $x^{-1} s x$ in $C_G(h)$. Finally, it is well-known (and easy to prove) that $\eta(y^{-1} h y) \equiv \eta(y^{-1} s y) \pmod{\mathfrak{p}}$ (see for instance [Bo, Proposition 2.14]). Therefore:

$$(\diamond) \quad \Psi_{g, \gamma}(V_{h, \eta}) \equiv \sum_{\substack{x \in [C_G(s) \backslash G / C_G(h)] \\ x h x^{-1} \in C_G(s)}} \eta(x^{-1} s x) \omega_{g, \gamma} \left(\sum_{\substack{y \in [C_G(g) \backslash C_G(s) x C_G(h) / C_G(h)] \\ y h y^{-1} \in C_G(g)}} \hat{\Sigma}_g(y h y^{-1}) \right) \pmod{\mathfrak{p}}.$$

Now, let $x \in [C_G(s) \backslash G / C_G(h)]$ be such that $x h x^{-1} \in C_G(s)$. Then, by definition of the Brauer map,

$$(\heartsuit) \quad \text{Br}_g^\theta(\hat{\Sigma}_s(x h x^{-1})) = \sum_{\substack{z \in [C_G(g) \backslash C_G(s) / (C_G(s) \cap C_G(x h x^{-1}))] \\ z(x h x^{-1}) z^{-1} \in C_G(g)}} \hat{\Sigma}_g((z x) h (z x)^{-1}).$$

But $(z x)_{z \in [C_G(g) \backslash C_G(s) / (C_G(s) \cap C_G(x h x^{-1}))]}$ is a set of representatives of double classes in $C_G(g) \backslash C_G(s) x C_G(h) / C_G(h)$. So it follows from (\diamond) and (\heartsuit) that

$$\Psi_{g, h}(V_{h, \eta}) \equiv \sum_{\substack{x \in [C_G(s) \backslash G / C_G(h)] \\ x h x^{-1} \in C_G(s)}} \eta(x^{-1} s x) \omega_{g, \gamma}(\text{Br}_g^\theta(\hat{\Sigma}_s(x h x^{-1}))).$$

Using now 2.5 and 2.6, we obtain that

$$\Psi_{g, h}(V_{h, \eta}) \equiv \Psi_{s, \sigma}(V_{h, \eta}) \pmod{\mathfrak{p}},$$

as desired. \square

Proof of Theorem 1.2. — Let (s, e) and (s', e') be two elements of $\mathcal{M}^p(G)$ such that $\mathcal{B}_G(s, e)$ and $\mathcal{B}_G(s', e')$ are contained in the same p -block of $\mathcal{M}(G)$ (see Proposition 2.11). Let $\sigma \in \text{Irr}(KC_G(s)e)$ and $\sigma' \in \text{Irr}(KC_G(s')e')$.

Then (s, σ) and (s', σ') are in the same p -block, so it follows from Proposition 2.9 that $s = s'$ and it follows from Proposition 2.10 that γ and γ' are in the same p -block of $C_G(s)$, that is $e = e'$. This completes the proof of Theorem 1.2. \square

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